

Levi Problem in Complex Manifolds

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Abstract

Let U be a pseudoconvex open set in a complex manifold M . When is U a Stein manifold? There are classical counter examples due to Grauert, even when U has real-analytic boundary or has strictly pseudoconvex points. We give new criteria for the Steinness of U and we analyze the obstructions. The main tool is the notion of Levi-currents. They are positive $\partial\bar{\partial}$ -closed currents T of bidimension $(1,1)$ and of mass 1 directed by the directions where all continuous psh functions in U have vanishing Levi-form. The extremal ones, are supported on the sets where all continuous psh functions are constant. We also construct under geometric conditions, bounded strictly psh exhaustion functions, and hence we obtain Donnelly-Fefferman weights. To any infinitesimally homogeneous manifold, we associate a foliation. The dynamics of the foliation determines the solution of the Levi-problem. Some of the results can be extended to the context of pseudoconvexity with respect to a Pfaff-system.

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1 Introduction

Let (M, ω) be a complex Hermitian manifold of dimension n . Let $U \Subset M$ be a relatively compact domain with smooth boundary. We can assume that $U := \{z \in U_1 : r(z) < 0\}$, where r is a function of class \mathcal{C}^∞ in a neighborhood U_1 of \bar{U} , such that dr is non-vanishing on ∂U . Recall that U is pseudoconvex if the Levi form of r is nonnegative on the complex tangent vectors to ∂U . More precisely,

$$\langle i\partial\bar{\partial}r(z), it \wedge \bar{t} \rangle \geq 0 \quad \text{if} \quad \langle \partial r(z), t \rangle = 0. \quad (1.1)$$

Condition (1.1) is independent of the choice of r .

The Levi problem is, whether a pseudoconvex domain is Stein, i.e., biholomorphic to a complex submanifold of \mathbb{C}^N , see [18].

Grauert has characterized Stein manifolds as follows.

Theorem 1.1. ([13]) *A complex manifold M is Stein iff there is a strictly psh exhaustion function on M .*

Narasimhan has given a similar characterization for Stein spaces [21].

The Levi problem admits a positive solution in many cases, in particular, when M is \mathbb{C}^n or \mathbb{P}^n . We refer to the surveys by Narasimhan [22], Siu [29], Peternell [25] and to the recent discussion by Ohsawa [24]. See also the book by Hörmander [18].

In the general case, Grauert has given two remarkable examples. Let $M := \mathbb{C}^n/\Lambda$ be a complex torus. Assume that $e_1 := (1, 0, \dots, 0)$ is the first vector in the lattice Λ . Let $\pi : \mathbb{C}^n \rightarrow M$ denote the canonical projection. Then $U := \pi(0 < \operatorname{Re} z_1 < 1/2)$ is pseudoconvex (Levi-flat) and U is not Stein. Indeed, $\pi(\operatorname{Re} z_1 = 1/4)$ is foliated by images of \mathbb{C} , hence holomorphic functions in U which are necessarily bounded on such images are constant.

Hirschowitz has analyzed such examples by introducing the notion of infinitesimally homogeneous manifolds. A manifold M is infinitesimally homogeneous if the global holomorphic vector fields generate the tangent space at every point of M . He then showed [16, 17]

Theorem 1.2. (Hirschowitz) *Let U be a domain in an infinitesimally homogeneous manifold. Assume U satisfies the *Kontinuitätssatz*. Then U admits a continuous psh exhaustion function. If moreover U does not contain a holomorphic image of \mathbb{C} , which is relatively compact in U , then U is Stein.*

A second example of Grauert [14] shows that the boundary of U can be strictly pseudoconvex except on a small set and still, U is not Stein. In the present article we analyze the obstructions of being Stein for pseudoconvex domains with smooth boundary. Our main tool is the notion of Levi currents. With the previous notations, a positive current T of bidimension $(1, 1)$ in M , supported on ∂U is a Levi current if it satisfies the following Pfaff system

$$T \wedge \partial r = 0, \quad T \wedge \partial \bar{\partial} r = 0, \quad i\partial \bar{\partial} T = 0, \quad \langle T, \omega \rangle = 1. \quad (1.2)$$

Observe that the support of the Levi current is very restricted, and that it is directed by the null space of the Levi form. We then obtain the following result.

Theorem 1.3. *Let $U \Subset M$ be a pseudoconvex domain with smooth boundary. If ∂U has no Levi current, then U is a modification of a Stein manifold. Moreover, there is a smooth function v , such that if $\rho := re^{-v}$, there is $\eta > 0$ such that the function $\hat{\rho} := -(-\rho)^\eta$ satisfies $i\partial \bar{\partial} \hat{\rho} \geq c|\hat{\rho}|\omega$ on $U \setminus K$, where K is compact.*

Clearly, when $M = \mathbb{C}^n$, or a Stein manifold, Levi currents do not exist. Indeed, positive currents with compact support, satisfying $i\partial \bar{\partial} T = 0$, are necessarily 0. So the last part of the above theorem is an extension of the Diederich-Fornæss theorem [3] which considers the case where $M = \mathbb{C}^n$. A crucial point in their proof

is that, for a pseudoconvex domain U in \mathbb{C}^n , the function $-\log \text{dist}(\cdot, \mathbb{C}^n \setminus \overline{U})$ is psh. This tool is not available here.

A similar result was proved when $M = \mathbb{P}^n$ or more generally for manifolds of positive holomorphic sectional curvature by Ohsawa and the author in [23]. It uses some geometric inequalities satisfied by the distance to the boundary due to Takeuchi and Elencwajg [32, 10].

The interest of constructing bounded exhaustion functions satisfying the above estimates is that the function $\psi := -\log(-\hat{\rho})$ satisfies the Donnelly-Fefferman condition and is proper and hence one can apply their theorem [7], see also [1].

When the domain U has real analytic boundary the non-existence of Levi currents is equivalent to the non-existence of a germ of holomorphic curve on the boundary of U . This uses results from [4].

In Section 2, after proving the above results, we address the question of finding bounded strictly psh exhaustion function $\hat{\rho}$, such that $\psi := -\log(-\hat{\rho})$ satisfies the Donnelly-Fefferman condition. We show in particular the following result.

Theorem 1.4. *Let $U \Subset M$ be a pseudoconvex domain with smooth boundary. Assume that there is a compact set $K \Subset U$ and a bounded function v on $U \setminus K$, such that $i\partial\bar{\partial}v \geq \omega$ on $U \setminus K$. Then U is a modification of a Stein manifold, and admits a bounded exhaustion function which is strictly psh out of a compact set.*

In Section 3 we give a criterion for Steinness. In Section 4 we introduce the notion of Levi currents on an arbitrary complex manifold (not just on the boundary of a pseudoconvex domain). So, to a non-Stein pseudoconvex domain, we associate a Pfaff system. The Levi currents correspond to generalized global solutions of that system. This permits to prove the following.

Theorem 1.5. *Let $U \Subset M$ be a pseudoconvex domain with smooth boundary. Assume it admits a continuous psh exhaustion function φ . Assume U is not a modification of a Stein manifold and that it contains at most finitely many compact varieties of positive dimension. Then there is a number t_0 , such that for every $t > t_0$ the level set $\{\varphi = t\}$ has a Levi-current T_t . In particular, each T_t is a positive $\partial\bar{\partial}$ -closed current of mass one with compact support.*

We also address briefly the general question of the existence of bounded strictly psh functions, through the notion of Liouville currents.

In Section 5, we show that if M is infinitesimally homogeneous and U is not Stein, then U is foliated by complex manifolds of fixed dimension $d > 0$, and the closure of each leaf is compact in U .

In Section 6 we give a foliated version of the above results. More precisely, we develop the notion of pseudoconvexity with respect to a Pfaff system.

2 Bounded psh exhaustion functions

The proof of Theorem 1.3 is based on the following proposition.

Proposition 2.1. *Let $U \Subset M$ be a pseudoconvex domain with smooth boundary. There is no Levi current on ∂U iff there is a smooth strictly psh function u in a neighborhood of ∂U .*

Proof. If u is a strictly psh function in a neighborhood of ∂U and T is a positive current supported on ∂U , then

$$\langle T, i\partial\bar{\partial}u \rangle = \langle i\partial\bar{\partial}T, u \rangle.$$

So if $i\partial\bar{\partial}T = 0$, we get that $T = 0$. Hence there is no Levi current.

We next show that any positive $\partial\bar{\partial}$ -closed current T of mass one supported on ∂U is a Levi current. Since T is $\partial\bar{\partial}$ -closed, then $\langle T, i\partial\bar{\partial}r^2 \rangle = 0$. Expanding and using that it is supported on $\{r = 0\}$, we get, $T \wedge i\partial r \wedge \bar{\partial}r = 0$. Therefore, $T \wedge \partial r = 0$.

Let χ be a smooth non-negative function with compact support. Using that $T \wedge \partial r = 0$, we get that:

$$0 = \langle T, i\partial\bar{\partial}(\chi r) \rangle = \langle T, \chi i\partial\bar{\partial}r \rangle.$$

But ∂U is pseudoconvex, i.e., $\langle i\partial\bar{\partial}r, it \wedge \bar{t} \rangle \geq 0$ when $\langle \partial r, t \rangle = 0$. The current T is directed by the complex tangent current space to ∂U because $T \wedge \partial r = 0$. It follows that $T \wedge \chi i\partial\bar{\partial}r = 0$ for an arbitrary χ . Hence, T is a Levi current on ∂U . So it is enough to show that if there is no $\partial\bar{\partial}$ -closed positive current of mass one supported on ∂U , there is a smooth strictly psh function in a neighborhood of ∂U .

Let

$$\mathcal{C} := \{T : T \geq 0 \text{ bidimension } (1,1) \text{ supported on } \partial U, \langle T, \omega \rangle = 1\},$$

and

$$Y := \{i\partial\bar{\partial}u, u \text{ test smooth function on } M\}^\perp.$$

The space Y is the space of the $i\partial\bar{\partial}$ -closed currents on M . We have assumed that $\mathcal{C} \cap Y$ is empty. The convex compact \mathcal{C} is in the dual of a reflexive space. The Hahn-Banach theorem implies that \mathcal{C} and Y are strongly separated. Hence, there is $\delta > 0$ and a test function u , such that $\langle i\partial\bar{\partial}u, T \rangle \geq \delta$, for every T in \mathcal{C} . So the function u , is strictly psh at all points of ∂U , and hence in a neighborhood of ∂U . Similar use of Hahn-Banach theorem occurs in [27, 31] \square

Since we have a strongly psh function in a neighborhood of ∂U , Theorem 1.3 will be a consequence of the following theorem.

Theorem 2.2. *Let K be a compact set in U . Assume there is function v in $U \setminus K$ such that one of the following conditions is satisfied:*

- (i) v is bounded and $i\partial\bar{\partial}v \geq \omega$;

(ii) $v \geq 0$, $i\partial v \wedge \bar{\partial} v \leq i\partial\bar{\partial}v$, and $i\partial\bar{\partial}v \geq \omega$.

Then there is a psh exhaustion function $\hat{\rho}$, vanishing on ∂U and such that on $U \setminus K$, $i\partial\bar{\partial}\hat{\rho} \geq |\hat{\rho}|\omega$.

Proof. We know that for $z \in \partial U$, $\langle i\partial\bar{\partial}r(z), it \wedge \bar{t} \rangle \geq 0$, when $\langle \partial r(z), t \rangle = 0$. It follows that there is a constant C , such that for $z \in \partial U$ and t arbitrary in the tangent space $T_z^{(1,0)}(M)$,

$$\langle i\partial\bar{\partial}r(z), it \wedge \bar{t} \rangle \geq -C|\langle \partial r(z), t \rangle||t|.$$

Choose a small neighborhood V of ∂U such that every point z in V projects to a point $z_1 \in \partial U$. Then $\partial r(z) = \partial r(z_1)O(r)$. Hence,

$$\begin{aligned} \langle i\partial\bar{\partial}r(z), it \wedge \bar{t} \rangle &\geq \langle i\partial\bar{\partial}r(z_1), it \wedge \bar{t} \rangle - C_0|r(z)||t|^2 \\ &\geq -C_1|\langle \partial r(z_1), t \rangle||t| - C_1|r(z)||t|^2 \\ &\geq -C|\langle \partial r(z), t \rangle||t| - C|r(z)||t|^2. \end{aligned} \quad (2.1)$$

Define $\rho := re^{-Av}$ and $\hat{\rho} := -(-\rho)^\eta$, we will choose A and η later. Observe first that by Richberg's theorem [26], we can assume that v is smooth. We write

$$\langle i\partial\bar{\partial}\hat{\rho}, it \wedge \bar{t} \rangle = \eta|r|^{\eta-2}e^{-A\eta v}[D(t)].$$

To get that $i\partial\bar{\partial}\hat{\rho} \gtrsim |\hat{\rho}|\omega$, we need to show that

$$[D(t)] \gtrsim |r|^2\langle \omega, it \wedge \bar{t} \rangle \quad (2.2)$$

If \mathcal{L} denotes $\langle i\partial\bar{\partial}r(z), it \wedge \bar{t} \rangle$, we have

$$\begin{aligned} D(t) &= Ar^2(\mathcal{L}v - \eta A|\langle \partial v, t \rangle|^2) + |r|(\mathcal{L}r - 2\eta \operatorname{Re}\langle \partial r, t \rangle \overline{\langle \partial v, t \rangle}) \\ &+ (1 - \eta)|\langle \partial r, t \rangle|^2. \end{aligned}$$

We also have

$$2\eta|r||\operatorname{Re}\langle \partial r, t \rangle \overline{\langle \partial v, t \rangle}| \leq r^2|\langle \partial v, t \rangle|^2 + \eta^2|\langle \partial r, t \rangle|^2.$$

So using relation (2.1) we get

$$\begin{aligned} D(t) &\geq Ar^2(\mathcal{L}v - (\eta A + A^{-1})|\langle \partial v, t \rangle|^2) + (1 - \eta - \eta^2)|\langle \partial r, t \rangle|^2 \\ &- C|r||\langle \partial r, t \rangle||t| - Cr^2|t|^2. \end{aligned}$$

Hence,

$$\begin{aligned} D(t) &\geq Ar^2\left(\mathcal{L}v - \frac{C}{A}|t|^2 - (\eta A + \frac{1}{A})|\langle \partial v, t \rangle|^2 - \frac{1}{A\sqrt{\eta}}|t|^2\right) \\ &+ (1 - \eta - \eta^2 - C\sqrt{\eta})|\langle \partial r, t \rangle|^2. \end{aligned}$$

Since $i\partial\bar{\partial}v \geq \omega$, and $i\partial v \wedge \bar{\partial} v \leq i\partial\bar{\partial}v$, it suffices to take $A \simeq \frac{1}{2\sqrt{\eta}}$ and η small enough. If v is bounded, we can assume $v \geq 0$ and replace v by Cv^2 , then condition (ii) is satisfied. \square

Remark 2.3. (i) In particular, we obtain that U is a modification of a Stein space.

(ii) Observe that when v extends smoothly to ∂U , as in Theorem 1.3, then $\hat{\rho}$ is Hölder continuous.

(iii) The conditions on v are of the type required for the Donnelly-Fefferman weights, except we do not ask for completeness i.e. that $v \rightarrow \infty$ when we approach ∂U .

Example 2.4. Let (M, ω) be a compact Kähler manifold. Let T be a positive closed current of bidegree $(1, 1)$, cohomologous to ω . Write $T - \omega = i\partial\bar{\partial}v$, we can assume $v \leq 0$. Assume that U is pseudoconvex and disjoint from the support of T . Then on U , we have $\omega = i\partial\bar{\partial}(-v)$. The hypothesis of Theorem 2.2 is satisfied if T admits locally bounded potentials. Otherwise the hypothesis of Theorem 3.1 below is satisfied.

Theorem 2.5. Let $U \Subset M$ be smooth pseudoconvex with real analytic boundary. Then ∂U has no Levi current iff it contains no germ of holomorphic curve.

Proof. Suppose ∂U has no Levi current. Then there is a smooth strictly psh function v near ∂U . Let W denote the union of non-trivial germs of holomorphic discs on ∂U and suppose W is nonempty. Consider the closure \bar{W} and let $p \in \bar{W}$ where the function v reaches its maximum on \bar{W} . According to the proof of Theorem 4 of [4] there is a point q close to p and a nontrivial subvariety V through q , in a polydisc centered at q of radius δ . Moreover, we can choose q arbitrarily close to p , without changing δ . Since v is strictly psh, we can assume that the maximum at p is reached at an interior point of V . A contradiction. So W is empty.

Assume now that ∂U does not have a non-trivial germ of holomorphic disc. It follows from Theorem 3 in [4] that the holomorphic dimension of any real analytic submanifold N is zero. This means that for every $z \in N$, $T_z^{(1,0)}(N)$ intersects

$$M_z := \{t : t \in T_z^{(1,0)}(\partial U), \langle i\partial\bar{\partial}r(z), it \wedge \bar{t} \rangle = 0\}$$

only at 0. So for each $t \in T_z^{(1,0)}(N)$, $\langle i\partial\bar{\partial}r(z), it \wedge \bar{t} \rangle > 0$. The authors in [4] state and prove their theorem in \mathbb{C}^n but this part of the argument is of local nature. Let N_0 denote the real analytic set of points $z \in \partial U$ where $\dim M_z > 0$. Then $N_0 \subset \bigcup_{k=1}^s N_k$, where N_k is a closed submanifold in $\partial U \setminus \bigcup_{j=1}^{k-1} N_j$, and $\langle i\partial\bar{\partial}r(z), it \wedge \bar{t} \rangle > 0$ for $t \in T_z^{(1,0)}(N_k)$. This follows from the Lojasiewicz stratification of real analytic sets and from the above statement, see [4]. Any Levi current T is a priori supported on N_0 .

Let ρ_j be a defining function of N_j and let χ be a cutoff function. If we expand $\langle T, i\partial\bar{\partial}(\chi\rho_j^2) \rangle = 0$, we get that $T \wedge \partial\rho_j = 0$.

Writing that $\langle T, i\partial\bar{\partial}(\chi\rho_j) \rangle = 0$, we get also that $T \wedge i\partial\bar{\partial}\rho_j = 0$. The non-degeneracy of $i\partial\bar{\partial}\rho_j$ on M_z implies that $T = 0$. \square

We recall the following form of a theorem by Donnelly-Fefferman, see [7] and [1].

Theorem 2.6. *Let N be a complex manifold of dimension n . Let $\Omega := i\partial\bar{\partial}\varphi$ be a complete Kähler metric on N . Assume there is $C_0 > 0$ such that $i\partial\varphi \wedge \bar{\partial}\varphi \leq C_0 i\partial\bar{\partial}\varphi$.*

Assume $p + q \neq n$. Then, for any (p, q) -form f in L^2 with $\bar{\partial}f = 0$, there is a solution u , to the equation $\bar{\partial}u = f$ with

$$\|u\|_{\Omega}^2 \leq C \|f\|_{\Omega}^2.$$

The condition on φ means just that $|d\varphi|_{\Omega}$ is bounded. The completeness means that $\varphi(z) \rightarrow \infty$ when $z \rightarrow \infty$ on N . The following proposition permits to apply the above theorem to the pseudoconvex domains considered previously. We just have to assume that U does not contain analytic varieties of positive dimension.

Proposition 2.7. *Let $U \Subset M$ be a pseudoconvex domain with a negative exhaustion function $\hat{\rho}$, satisfying*

$$i\partial\bar{\partial}\hat{\rho} \gtrsim |\hat{\rho}|\omega.$$

Let $\varphi := -\log(-\hat{\rho})$. Then the metric $\Omega := i\partial\bar{\partial}\varphi$ is complete and $|d\varphi|_{\Omega}$ is bounded. So Theorem 2.6 applies.

Proof. We have

$$i\partial\bar{\partial}\varphi = \frac{i\partial\bar{\partial}\hat{\rho}}{|\hat{\rho}|} + \frac{i\partial\hat{\rho} \wedge \bar{\partial}\hat{\rho}}{\hat{\rho}^2} = \frac{i\partial\bar{\partial}\hat{\rho}}{\hat{\rho}} + i\partial\varphi \wedge \bar{\partial}\varphi \geq c\omega + i\partial\varphi \wedge \bar{\partial}\varphi.$$

Moreover, since $\hat{\rho} \rightarrow 0$ when $z \rightarrow \partial U$, the metric Ω is complete. \square

Remark 2.8. *In Theorem 2.2, we start with a potential v satisfying $i\partial v \wedge \bar{\partial}v \leq i\partial\bar{\partial}v$, but the metric is not necessarily complete. We end up with a complete one associated to $\varphi := -\log(-\hat{\rho})$.*

3 A condition for Steiness of a pseudoconvex domain

As recalled in the introduction, according to Grauert's theorem, to prove that a pseudoconvex domain is Stein, one should construct a strictly psh exhaustion function. Here we give a quite weak assumption in order to construct such an exhaustion.

Theorem 3.1. *Let $U \Subset M$ be a pseudoconvex domain with smooth boundary. Assume there is a neighborhood V of ∂U , and a function v on $U \cap V$ such that the following conditions are satisfied:*

(i) $i\partial\bar{\partial}v \geq \omega$;

(ii) If r denotes a defining function for ∂U , then for any $\epsilon > 0$, $v > \epsilon \log |r|$ when $r \rightarrow 0$.

Then U admits a bounded exhaustion function, with all level sets strictly pseudoconvex. Moreover, U is a modification of a Stein space.

When v is defined near ∂U and satisfies $i\partial\bar{\partial}v \geq \omega$, Elencwajg [10] showed that U is a modification of a Stein manifold (see also [29]).

Proof. Define $\sigma := re^{-Av}$. According to Richberg's approximation theorem [26] we can assume that v is smooth. Condition (ii) implies that for every $A > 0$, σ is an exhaustion function. We have $\bar{\partial}\sigma = e^{-Av}(\bar{\partial}r - A\bar{\partial}v)$ and

$$i\partial\bar{\partial}\sigma = e^{-Av} \left(i\partial\bar{\partial}r - 2A\operatorname{Re}(i\partial r \wedge \bar{\partial}v) - A\bar{\partial}r \wedge \bar{\partial}v + A^2 r \partial v \wedge \bar{\partial}v \right).$$

We are going to check that the level sets of σ are strictly pseudoconvex. If t is a $(1, 0)$ tangent vector to a level set, then $\langle \partial\sigma, t \rangle = 0$ i.e. $\langle \partial r, t \rangle = A\bar{r}\langle \partial v, t \rangle$. So

$$\begin{aligned} \langle i\partial\bar{\partial}\sigma, it \wedge \bar{t} \rangle &= e^{-Av} \left(\langle i\partial\bar{\partial}r, it \wedge \bar{t} \rangle + 2A^2 |r| |\langle \partial v, t \rangle|^2 \right. \\ &\quad \left. + A^2 r |\langle \partial v, t \rangle|^2 + A|r| \langle i\partial\bar{\partial}v, it \wedge \bar{t} \rangle \right). \end{aligned}$$

We also have near ∂U that

$$\langle i\partial\bar{\partial}r(z), it \wedge \bar{t} \rangle \geq -C(|\langle \partial r(z), t \rangle| |t| + |t|^2).$$

So if $A > C$, and A is large enough.

$$\begin{aligned} e^{Av} \langle i\partial\bar{\partial}\sigma, it \wedge \bar{t} \rangle &= -CA|r| |\langle \partial v, t \rangle|^2 - 2C|r| |t|^2 \\ &\quad + A^2 |r| |\langle \partial v, t \rangle|^2 + A|r| \langle i\partial\bar{\partial}v, it \wedge \bar{t} \rangle \\ &\geq |r| (A \langle i\partial\bar{\partial}v, it \wedge \bar{t} \rangle - 2C|t|^2) > 0. \end{aligned}$$

It follows that there is a function $C(\sigma)$ such that

$$i\partial\bar{\partial}\sigma \geq -\frac{C(\sigma)}{2} i\partial\sigma \wedge \bar{\partial}\sigma \quad \text{near} \quad \partial U.$$

Let $\kappa(t) := \int_{-1}^t C(s)ds$, and $\chi(t) := \int_{-1}^t e^{\kappa(s)} ds$. Then $\chi'' - C(s)\chi'(s) = 0$. Define $\rho := \chi(\sigma)$. Then ρ is a strictly psh exhaustion function. Indeed,

$$\begin{aligned} i\partial\bar{\partial}\rho &= \chi'(\sigma) i\partial\bar{\partial}\sigma + \chi''(\sigma) i\partial\sigma \wedge \bar{\partial}\sigma \\ &> \left(\frac{-C}{2} \chi'(\sigma) + \chi''(\sigma) \right) i\partial\sigma \wedge \bar{\partial}\sigma \geq \frac{C}{2} \chi'(\sigma) i\partial\sigma \wedge \bar{\partial}\sigma. \end{aligned}$$

On the other hand, when $\langle \partial\sigma, t \rangle = 0$, we also have strict positivity. So ρ is a strictly psh exhaustion function. \square

4 An obstruction to Steiness: Levi currents

Let $U \Subset M$ be a locally Stein domain in a complex Hermitian manifold (M, ω) . It is not clear whether there are non-constant psh functions in U .

When U admits a continuous psh exhaustion function, the domain U may not have strictly psh functions and hence is not necessarily Stein, this is the case in Grauert examples [13, 14], or in the families described by Ohsawa [24]. When M is infinitesimally homogeneous manifold [16, 17], the domain U has a psh exhaustion function, but may not have strictly psh functions and hence is not necessarily Stein.

In this section we want to discuss an obstruction to Steiness given by Levi currents with compact support in U . In order to introduce the notion we need to define $T \wedge i\partial\bar{\partial}v$, when T is a positive current $\partial\bar{\partial}$ -closed and v is a continuous psh function. We recall few results from [8].

Let T be a positive current of bidegree (p, p) . Assume that $i\partial\bar{\partial}T$ is of order 0. When T is a current of order 0, the mass of T on a compact K is denoted by $\|T\|_K$. When T is positive, and M is of dimension n , $\|T\|_K$ is equivalent to $|\int_K T \wedge \omega^{n-p}|$.

When u is a smooth psh function on an open set $V \subset M$ we have

$$i\partial\bar{\partial}u \wedge T := u(i\partial\bar{\partial}T) - i\partial\bar{\partial}(uT) + i\partial(\bar{\partial}u \wedge T) - i\bar{\partial}(\partial u \wedge T). \quad (4.1)$$

The following estimates are proved in [8]. Let $L \Subset K$ be two compact sets in V . Assume T is positive and $i\partial\bar{\partial}T$ is of order 0. Then there is a constant $C_{K,L} > 0$ such that for every smooth bounded psh function u on V , we have

$$\int_L i\partial u \wedge \bar{\partial}u \wedge T \wedge \omega^{n-p-1} \leq C_{K,L} \|u\|_{L^\infty(K)}^2 (\|T\|_K + \|i\partial\bar{\partial}T\|_K) \quad (4.2)$$

and

$$\|i\partial\bar{\partial}u \wedge T\|_L \leq C_{K,L} \|u\|_{L^\infty(K)} (\|T\|_K + \|i\partial\bar{\partial}T\|_K). \quad (4.3)$$

This permits to extend relation (4.1) to u psh and locally bounded. Moreover, when u_n converges locally uniformly to u then

$$I_{n,m} := \int_L i\partial(u_n - u_m) \wedge \bar{\partial}(u_n - u_m) \wedge T \wedge \omega^{k-p-1}$$

converges to 0. It is enough to prove that for a ball B and to assume u_n and u_m coincide near the boundary of B , see [8]. So

$$I_{n,m} = \frac{-1}{2} \int (u_n - u_m) i\partial\bar{\partial}(u_n - u_m) \wedge T + \frac{1}{2} \int i\partial\bar{\partial}(u_n - u_m)^2 \wedge T.$$

Hence,

$$\begin{aligned} 2I_{n,m} &\leq \|u_n - u_m\|_B^2 \|i\partial\bar{\partial}T\|_B + \int_B |u_n - u_m| i\partial\bar{\partial}u_n \wedge T + \int_B |u_n - u_m| i\partial\bar{\partial}u_m \wedge T \\ &\quad + \int_B |u_n - u_m| i\partial\bar{\partial}u_n \wedge T. \end{aligned}$$

The convergence follows using (4.3).

Estimate (4.2) permits also to define $\partial u \wedge T$. Then $\partial u_n \wedge T \rightarrow \partial u \wedge T$, as currents of order 0, if u_n converges to u uniformly on compact sets.

Definition 4.1. *A Levi current in V is a nonzero positive current of bidimension $(1, 1)$, such that $i\partial\bar{\partial}T = 0$ and $T \wedge i\partial\bar{\partial}v = 0$ for every continuous psh function in V .*

A Liouville current in V is a nonzero positive current of bidimension $(1, 1)$, such that $i\partial\bar{\partial}T = 0$ and $T \wedge i\partial\bar{\partial}v = 0$ for every bounded continuous psh function in V .

Observe that this implies that for a Levi current T (resp. a Liouville current) we have that $T \wedge \bar{\partial}v = 0$ for every continuous psh function v , (resp. for a bounded continuous psh function v).

Denote by $\mathcal{L}(V)$ (resp $\mathcal{A}(V)$), the space of Levi currents (resp. Ahlfors currents) on V .

Proposition 4.2. *$\mathcal{L}(V)$ (resp $\mathcal{A}(V)$), is closed for the weak topology of currents. There is a Levi current T_L with support F_L (resp. a Liouville current T_A , with support F_A), such that all Levi currents are supported on F_L (resp. all Liouville currents are supported on F_A).*

The closed sets F_L and F_A , satisfy the local maximum principle for psh functions near F_L (resp. F_A).

Proof. We just consider the case of Levi currents, the case of Liouville currents is similar. Let T_n , be a sequence of Levi currents converging to the positive current T . It is clear that $\partial\bar{\partial}T = 0$. For any cutoff function χ with small support, we have that $\chi T_n \wedge i\partial\bar{\partial}v$ converge to $\chi T \wedge i\partial\bar{\partial}v$, for v smooth, hence for v psh continuous, as it follows from our preliminary study. Hence the current T , is in $\mathcal{L}(V)$. So $\mathcal{L}(V)$ is closed, hence separable i.e. it contains a countable dense set (T_j) . If ϵ_n , are small enough, we can define $T_L := \sum \epsilon_n T_n$. Define F_L as the support of T_L . It follows from [27], that it satisfies the local maximum principle for functions psh near F_L . □

Proposition 4.3. *Let K be a compact set in V . If S is a positive current of bidimension $(1, 1)$, supported in K , such that $i\partial\bar{\partial}S = 0$, then S is a Levi current. The convex set $\mathcal{L}(K)$ of Levi currents of mass 1, supported in K is compact.*

Let T be a Levi current in V . Let v be a non-negative continuous psh function, then the current vT , is a Levi current. If T is an extremal Levi current in V , then continuous psh functions in V , are constant on $H := \text{supp}(T)$.

A similar statement holds for Liouville currents in V .

Proof. Suppose S is a positive current of bidimension $(1, 1)$, $\partial\bar{\partial}$ -closed and supported on K . We observe first that for u continuous and psh in a neighborhood

of K , $\bar{\partial}u \wedge S$, is well defined and of order 0. Then we apply (4.1) to the function 1. This shows that $\partial\bar{\partial}u \wedge S = 0$. So S is a Levi current. It follows that $\mathcal{L}(K)$ is compact.

Assume T is a Levi current in V . Let v be a continuous psh function in V . Let h be a convex strictly increasing function. Since $\partial\bar{\partial}h(v) \wedge T = 0$, we get that $\bar{\partial}v \wedge T = 0$. We then apply formula (4.1) and get that $-\partial\bar{\partial}(vT) = \partial\bar{\partial}v \wedge T = 0$.

Assume T is extremal. Let u be continuous psh in V . Suppose $(u < 0)$ and $(u > 0)$ are two nonempty open sets in H . Let χ be a convex increasing function vanishing for $t < 0$ and strictly increasing for $t > 0$. Then the current $S := \chi(u)T$ is a Levi current as we have seen. This contradicts the extremality of T .

The proof for Liouville currents is similar.

□

Theorem 4.4. *Let N be a complex manifold with a psh continuous exhaustion φ . Then N is Stein iff there is no Levi current with compact support in N .*

If N admits a bounded continuous psh exhaustion function, then it admits a bounded strictly psh exhaustion function iff there is no Levi current with compact support in N .

Proof. If there is a strictly psh, continuous function v in N and T is a positive current such that $T \wedge \partial\bar{\partial}v = 0$, then $T = 0$. We have to prove the converse. We show that if there is no Levi current on a compact set K , then there is a smooth function v_K , strictly psh in a neighborhood of K .

Suppose S is a positive current of bidimension $(1, 1)$ $\partial\bar{\partial}$ -closed and supported on K . As we have seen, for every u continuous psh near K , in particular, for u continuous psh on N , we have that $S \wedge i\partial\bar{\partial}u = 0$. So S is a Levi current. Hence $S = 0$. The duality argument used in the proof of Proposition 2.1 implies the existence of v_K smooth and strictly psh near K .

For a compact K let \hat{K} denote the hull with respect to continuous psh functions. Since there is a psh continuous exhaustion function we can choose $K_n \nearrow N$, $K_{n+1} \Subset K_n$, and $K_n = \hat{K}_n$. Let v_n be a continuous function strictly psh near K_{n+1} . Let χ_n be a convex increasing function such that

$$\chi_n(\varphi) < \inf_{K_n} v_n \quad \text{on } K_{n-1}. \quad \text{and} \quad \chi_n(\varphi) > \sup v_n \quad \text{near } \partial K_n.$$

Define

$$u_n := \sup(\chi_n(\varphi), v_n).$$

Then u_n is psh continuous on N and strictly psh near K_{n-1} . Moreover, it is an exhaustion.

If we choose $0 < \epsilon_n < \frac{1}{2^n} \|u_n\|_{K_n}^{-1}$, then the function $u := \sum_n \epsilon_n u_n$ is strictly psh function. So the function $u + \chi(\varphi)$ is strictly psh and an exhaustion if χ is convex increasing fast enough.

Suppose now that $\varphi < 0$ psh and $\varphi \rightarrow 0$ when $z \rightarrow \infty$ on N . We consider u_n as above and define $w_n := \epsilon_n(u_n - c_n)$ with $c_n := \lim_{z \rightarrow \infty} u_n(z)$. It is clear

that c_n is constant. If ϵ_n is small enough, $w := \sum w_n$ is a bounded strictly psh exhaustion. \square

Proof of Theorem 1.5. We will need the following theorem of Grauert. If N is a complex manifold with a continuous psh exhaustion φ , such that φ is strictly psh out of a compact set K of N , then N is a proper modification of a Stein space. More precisely, one can blow down analytic sets in N to points and get holomorphic convexity for compact sets in the blow down.

Suppose N is not a modification of a Stein space and that for a sequence $t_n \rightarrow \infty$, there is no Levi current on $(\varphi = t_n)$. We have seen that this implies the existence of a smooth strictly psh function near $(\varphi = t_n)$. Using the construction in the previous theorem, there is a continuous psh exhaustion ψ , strictly psh in a neighborhood of each $(\varphi = t_n)$. So $(\varphi < t_n)$ is a modification of a Stein space. In particular the compact analytic sets A_j are necessarily in $(\varphi = s_j)$, for some s_j . From the finiteness assumption the A_j cannot accumulate near the boundary. So the s_j are uniformly bounded. Hence, there is t_1 such that for $t > t_1$, there is no Levi current with compact support on $t > t_1$. Here again, we use Grauert's Theorem. The argument in Theorem 4.4 shows that we can construct a psh exhaustion function strictly psh on $(\varphi > t_1)$. Hence U is a modification of a Stein space. A contradiction. So, for t large enough there is a Levi current on $(\varphi = t)$. \square

The following proposition describes the function theory near the support of a $\partial\bar{\partial}$ -closed in \mathbb{P}^k .

Proposition 4.5. *Let T be an extremal positive current of bidimension $(1, 1)$, $\partial\bar{\partial}$ -closed in \mathbb{P}^k with support K . Then there exists a fundamental sequence of open neighborhoods (U_n) of K , such that every psh function in U_n is constant.*

Proof. Let u be a psh function in a neighborhood V of K . Since \mathbb{P}^k is homogeneous, we can assume that u is smooth and satisfies $0 < u < 1$. As we have seen T is a Levi current, hence uT is $\partial\bar{\partial}$ -closed. The extremality of T implies that u is constant on K . We can consider, the images of T by automorphisms of \mathbb{P}^k , close to identity with a fixed point in K . The function u has to be constant on the images of K . The theorem follows. \square

Remark 4.6. 1) Grauert's example shows that we cannot replace in the previous statement the projectif space by a torus.

2) Similarly, if K is a minimal compact laminated set in \mathbb{P}^k , with finitely many singular points. Then one can construct a fundamental sequence (U_n) of open neighborhoods of K , such that every psh function in U_n is constant. For basics on laminations see for example the survey [12].

3) The complement of the support of a positive current of bidimension $(1, 1)$ and $\partial\bar{\partial}$ -closed is 1-pseudoconvex (in dimension 2 it is pseudoconvex). So the

support is quite large see [11, 28]. The support H , of a positive $\partial\bar{\partial}$ -closed current satisfies the local maximum principle for psh functions near H , see [27].

4) If U is not Stein but admits a continuous psh exhaustion function, the classes

$$C_a := \{z : u(z) = u(a) \text{ for every } u \text{ psh continuous}\}$$

are nontrivial and indeed are on $(\varphi = t)$, hence some sets C_a are of Hausdorff dimension larger or equal to 2. Finally, the extremal Levi currents in U , are Levi currents on $(\varphi = t)$ except the level set is not necessarily smooth.

We next address the relation with pseudoconcave manifolds. We first recall some definitions, see [15]

Definition 4.7. *A real C^2 function u defined in a complex manifold U is strictly q -convex if the complex Hessian (Levi form) has at least q strictly positive eigenvalues at every point. A complex manifold U is q -complete if it admits a strictly q -convex exhaustion function.*

Definition 4.8. *A function ρ is strictly q -convex with corners on U if for every point $p \in U$ there is a neighborhood U_p and finitely many strictly q -convex C^2 functions $\{\rho_{p,j}\}_{j \leq \ell_p}$ on U_p such that $\rho|_{U_p} = \max_{j \leq \ell_p} \{\rho_{p,j}\}$. The manifold U is strictly q -complete with corners if it admits an exhaustion function which is strictly q -convex with corners.*

Theorem 4.9. *Let K be a compact set in a connected complex manifold U . Assume $U \setminus K$, is strictly 2-complete with corners. Then $U \setminus K$, admits no non-constant psh function. Moreover there is a non-zero, positive $\partial\bar{\partial}$ -closed current T , of bidimension $(1,1)$ supported in K .*

Proof. Let ρ denote the strictly 2-convex exhaustion function with corners. For each p in a level set $(\rho = c)$, there is on a neighborhood U_p , a strictly 2-convex function ρ_p , such that $\rho_p(p) = c$, and $\rho \geq \rho_p$ in U_p . If at p the gradient of ρ_p is non zero there is still a strictly positive eigenvalue of the Levi-form in the tangent space. Using the Taylor expansion at p one sees easily that, there is a holomorphic disc through p and otherwise contained in $(\rho \geq c)$, which enters in $(\rho > c)$, see [18] p.51. If the gradient of ρ_p vanishes at p , the construction of an analytic disc with the above properties is even simpler. Hence if u is psh in $(\rho > c)$, by maximum principle, u is constant on each component of $(\rho > c)$. Since c is arbitrary and U is connected the result follows.

In particular there are no strictly psh functions near K . The existence of T follows from the duality principle we have already used. \square

We end up this section with few remarks on bounded psh functions.

For a non-compact connected Riemann surface N , the existence of a non-constant bounded subharmonic function is equivalent to the existence of a Green function with a pole at a point p in N . The notion seems much less explored in

several complex variables. We give few remarks on the question. We introduce first the following definition.

Definition 4.10. *A connected complex manifold N is Ahlfors hyperbolic iff it admits a smooth bounded strictly psh function.*

Using Richberg's approximation Theorem [26], this is equivalent to the existence of a continuous bounded strictly psh function. It is clear that such manifold does not have non-zero Liouville currents. We give a class of examples.

Let \mathbb{P}^k denote the complex projective space of dimension k . Consider an endomorphism $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$ which is holomorphic and of algebraic degree d strictly larger than 1. Let ω denote the Fubini-Study form on \mathbb{P}^k . The Green current T associated to f is given by: $T = \lim_n (d^{-n} (f^n)^*(\omega)) = \omega + i\partial\bar{\partial}g$. The function g is Hölder continuous.

Moreover the complement of $\text{supp}(T)$ is the Fatou set, [9]. So any component U of the Fatou set is Ahlfors hyperbolic.

More generally, let (M, ω) be a compact Kähler manifold. Let T be a positive closed current of bidegree $(1, 1)$ cohomologous to ω , with locally bounded potentials. Then the components of the complement of $\text{supp}(T)$, are Ahlfors hyperbolic.

Let a be a positive irrational number. Consider in \mathbb{C}^2 the domain

$$U_a := \{(z, w) / |w||z|^a < 1\}.$$

This domain is Stein, but is not an Ahlfors hyperbolic domain.

Indeed any bounded psh function is constant on the level sets of the function $u(z, w) = |w||z|^a$. It is easy to see that each such level sets supports a non zero positive closed Liouville current T . The current is unique up to a multiplicative constant. On the level set $(u = c)$, the current is given by, $T_c = i\partial\bar{\partial}(\log(\sup(u, c)))$.

In the definition we have asked for the functions to be smooth to avoid examples like the following. Let φ be a subharmonic function in \mathbb{C} , taking the value $-\infty$, on a dense set. Define

$$U := \{(z, w) / |w| \exp(\varphi(z)) < 1\}.$$

Then U admits non-constant bounded psh functions, but every continuous bounded psh function is constant.

In [23] there is an example of a Stein domain U with smooth boundary, relatively compact in a homogeneous manifold M , such that all bounded psh functions in U are constant. Indeed U is foliated by images of \mathbb{C} which cluster on the boundary.

Proposition 4.11. *Let (M, ω) , be a compact Kähler manifold. Let $U \subset M$, be a domain, with a non-constant continuous bounded psh function u defined in U , reaching its minimum c in U . Let $X_c := (z \in U, u(z) = c)$. Then, either $\overline{X_c}$ supports a non-zero positive $\partial\bar{\partial}$ -closed current, or there is in U , a bounded continuous psh function v , which is strictly psh in a neighborhood of X_c .*

Proof. Suppose there is no non-zero positive $\partial\bar{\partial}$ -closed current, supported on X_c . Then, there is a strictly psh function w , in a neighborhood W of $\overline{X_c}$. We can assume that $c = 0$, and that on W , $0 < w < 1$. Composing with a convex, increasing function, we can assume that out of W we have $u > 1$. It suffices to define $v = \sup(u, w)$. The function is well defined in U , and is strictly psh near X_c . \square

5 Manifolds with holomorphic vector fields

In this section we discuss the Levi problem on manifolds M , on which the space \mathcal{V} of holomorphic vector fields is of positive dimension. Hirschowitz has considered manifolds M on which at every point p , \mathcal{V} generates the tangent space of M at p . He called such manifolds infinitesimally homogeneous. When $\dim \mathcal{V} \geq 1$, we will say that M is partially infinitesimally homogeneous.

Let D denote the open unit disc in \mathbb{C} . Recall that a domain U in a complex manifold M satisfies the *Kontinuitätssatz* if the following holds. For any sequence $f_j : \overline{D} \rightarrow U$ of holomorphic maps on D , continuous on \overline{D} , such that $\bigcup_j f_j(\partial D)$ is relatively compact in U , then $\bigcup_j f_j(\overline{D})$ is relatively compact in U .

Hirschowitz showed that if U satisfies the *Kontinuitätssatz* in an infinitesimally homogeneous manifold, then U admits a continuous psh exhaustion function. We first refine his result, then we describe the pseudoconvex non-Stein domains in an infinitesimally homogeneous manifold. They carry a holomorphic foliation with very special dynamics.

Theorem 5.1. *Let $U \subset M$ be a domain satisfying the *Kontinuitätssatz*. Assume that at each point $p \in \partial U$, there is $Z \in \mathcal{V}$ transverse to the boundary at p , we will write $\partial U \pitchfork \mathcal{V}$. Then U admits a continuous psh exhaustion function φ .*

Proof. A vector field Z is transverse at p to ∂U , if the local solution of Z around p passes through U and through $M \setminus \overline{U}$. For a vector field Z in M , we consider the flow $g_Z(z, \zeta)$, such that $g_Z(z, 0) = z$. Let Ω_z denote the connected component in \mathbb{C} containing 0, of the open set $(\zeta \in \mathbb{C}, \quad g_Z(z, \zeta) \in U)$. Let

$$U_Z := \{(z, \zeta) : z \in M, \zeta \in \mathbb{C}, \zeta \in \Omega_z\}.$$

We define $d_Z(z)$ as the distance of z to the boundary along the vector field Z . More precisely,

$$d_Z(z) := \sup\{|\zeta| : (z, \zeta) \in U_Z\}.$$

Lemma 5.2. *If U is not invariant under the flow φ_Z , then $-\log d_Z$ is psh on U .*

Proof. We observe that if the domain U_Z satisfies the *Kontinuitätssatz*, it will follow that $-\log$ distance to the complement in the ζ -direction is psh. When Z is transverse to the boundary at some point, the function is not identically $-\infty$.

Let $f_j : \overline{D} \rightarrow U_Z$. Assume $\bigcup_j f_j(\partial D) \Subset U_Z$. Let $\pi : U_Z \rightarrow U$ be the projection. Since U satisfies the Kontinuitätssatz, $\bigcup_j \pi \circ f_j(\overline{D}) \Subset U$. That we have also the compactness in the ζ -direction follows from the standard results on solutions of vector fields. \square

End of the proof of Theorem 5.1. When U is relatively compact, we just need finitely many vector fields \mathcal{V} , such that for every point $p \in \partial U$, there is $Z \in \mathcal{V}$, transverse to ∂U at p . Then the function $z \rightarrow \sup\{-\log d_Z(z) : Z \in \mathcal{V}\}$ is a continuous psh exhaustion. When \overline{U} is not compact, the construction can be easily adapted, to get the continuity. Indeed, we need only finitely many vector fields on each compact of \overline{U} . \square

Remark 5.3. When M is infinitesimally homogeneous, the hypothesis is always satisfied. Otherwise, one should observe that if it is satisfied for U , it holds also for domains close enough to U , in the \mathcal{C}^1 -topology.

Let v be a continuous subharmonic function in the unit disc D . We will say that v is strictly subharmonic at 0 if the Laplacean $\Delta v > 0$ in a neighborhood of 0. This is equivalent to the fact that small \mathcal{C}^2 perturbations of v in a neighborhood of 0, are still subharmonic. For a continuous psh function u in U , we will write $\langle i\partial\bar{\partial}u(z), it_z \wedge \bar{t}_z \rangle = 0$ iff $u \circ f$ is not strictly subharmonic at 0, for a holomorphic map $f : D \rightarrow U$, $f(0) = z$, $f'(0) = t_z$. Let $\mathcal{P}(U)$ denote the continuous psh functions on U . For a compact set $K \Subset U$, let $\mathcal{P}(K)$ denote the cone of psh continuous near K . We define

$$\mathcal{N}_z(K) := \{t_z : t_z \in T^{1,0}(U), \langle i\partial\bar{\partial}u(z), it_z \wedge \bar{t}_z \rangle = 0 \text{ for } u \in \mathcal{P}(K)\}.$$

Denote by $\mathcal{N}(K) := \bigcup_{z \in U} \mathcal{N}_z(K)$. The set $\mathcal{N}(K)$ is closed in the tangent bundle on K . Let $\mathcal{N}_z := \bigcap_K \mathcal{N}_z(K)$ as $K \nearrow U$, and $\mathcal{N} := \bigcup \mathcal{N}_z$.

Proposition 5.4. *Suppose there is $Z \in \mathcal{V}$, with $Z(z_0) = t_{z_0} \in \mathcal{N}_{z_0}$. Then the orbit of Z , starting at z_0 is contained in \mathcal{N} . If moreover $\partial U \pitchfork \mathcal{V}$, then the orbit is complete and is contained in a level set of any psh continuous function $u \in \mathcal{P}(U)$, in particular in the level sets of the exhaustion φ .*

Proof. Let g denote the complex flow of Z . Then for $u \in \mathcal{P}(K)$, $u \circ g(z_0, \zeta)$ is also in $\mathcal{P}(K)$, if ζ is small enough. Since g is a local biholomorphism, $g(z_0, \zeta)(t_{z_0}) \in \mathcal{N}_{g(z_0, \zeta)}$.

We can approximate functions in $\mathcal{P}(K)$ by functions in $\mathcal{P}(K)$, smooth along the orbits of \mathcal{V} . Let B be a neighborhood of 0 in \mathcal{V} and let D denote the unit disc in \mathbb{C} . Let ρ be an approximation of the identity in $B \times D$. It suffices to consider the approximation

$$\langle \rho(Z, \zeta), u(g_Z(z, \zeta)) \rangle.$$

We can use functions smooth on orbits. If $Z \in \mathcal{N}$, we get that $\langle \bar{\partial}u, Z \rangle = 0$. Indeed, we can apply the definition to $\exp(u)$. So u is constant along the orbit

of Z . Since φ is an exhaustion, the orbit is contained in a compact level set of φ and hence we have a holomorphic image of \mathbb{C} in that level set. \square

Theorem 5.5. *Suppose M is infinitesimally homogeneous. Let $U \subset M$ satisfy the Kontinuitätssatz. There is an integer $0 \leq d < n$, and a foliation \mathcal{F} , with leaves of dimension d on the level sets of the exhaustion φ . If $d = 0$, then U is Stein. If $d \geq 1$, then $(i\partial\bar{\partial}\varphi)^{n+d-1} = 0$ and there is a positive closed Levi current T_t , of bidimension (d, d) and mass one, on each level set $(\varphi = t)$, for $t \geq t_0$.*

When $U \Subset M$ and U is non Stein there is a positive closed Levi current S of bidimension (d, d) on ∂U .

Lemma 5.6. *The bundle \mathcal{N} , is of constant rank d , with $0 \leq d < n$. The sections are stable under Lie bracket.*

Proof. Since M is infinitesimally homogeneous, for each non-zero vector $t_z \in \mathcal{N}_z$ there is a holomorphic vector field Z in that direction. So we can apply Proposition 5.4. Moreover, the image by the flow g of \mathcal{N} is contained in \mathcal{N} . So the support of \mathcal{N} is U . We show that \mathcal{N} is stable under Lie bracket.

In an infinitely homogeneous manifold, continuous psh functions near K are approximable by smooth ones [17]. So to analyze \mathcal{N}_K , we can use smooth psh functions.

Let X, Y and Z be $(1, 0)$ -holomorphic vector fields. If $X, Y \in \mathcal{N}(K)$, then $\langle \bar{\partial}u, X \rangle = \langle \bar{\partial}u, Y \rangle = 0$ and $\langle \partial\bar{\partial}u, X \wedge \bar{Z} \rangle = 0$ for every $Z \in \mathcal{V}$, and for any smooth psh function in a neighborhood of K . We also have for type reasons:

$$\langle \partial\bar{\partial}u, [X, Y] \wedge \bar{Z} \rangle = -\langle \partial u, [[X, Y], \bar{Z}] \rangle.$$

Jacobi's identity gives that

$$[[X, Y], \bar{Z}] = [X, [Y, \bar{Z}]] - [Y, [X, \bar{Z}]].$$

So if $X, Y \in \mathcal{N}(K)$, i.e. $\langle \partial\bar{\partial}u, X \wedge \bar{X} \rangle = \langle \partial\bar{\partial}u, Y \wedge \bar{Y} \rangle = 0$, then

$$\langle \partial\bar{\partial}u, X \wedge \bar{Z} \rangle = \langle \partial\bar{\partial}u, Y \wedge \bar{Z} \rangle = 0$$

for every $Z \in \mathcal{V}$. It follows that $[X, Y] \in \mathcal{N}(K)$. \square

Proof of Theorem 5.5. It is clear that the rank of $\mathcal{N}(K)$ decreases as K increases. Hence it stabilizes. So \mathcal{N} is a bundle of dimension d , stable under Lie bracket. We hence get a foliation \mathcal{F} . Moreover for every $Z \in \mathcal{N}$ we have that $\langle \partial u, Z \rangle = 0$. Hence the leaves are contained in the level sets of functions in $\mathcal{P}(U)$. In particular the leaves are contained in $(\varphi = \text{const})$.

Since there is an exhaustion function, necessarily $d < n$. When $d = 0$, it is easy to construct a strictly psh exhaustion function

Any continuous psh function u , is constant on the leaves, hence $(i\partial\bar{\partial}u)^{n-d+1} = 0$. In particular, φ is constant on leaves, hence $(i\partial\bar{\partial}\varphi)^{n-d+1} = 0$. We can replace φ by $\exp(\varphi)$, so it follows that $i\partial\varphi \wedge \bar{\partial}\varphi \wedge (i\partial\bar{\partial}\varphi)^{n-d} = 0$. Consequently, for every nonnegative function χ , the current $\chi(\varphi)(i\partial\bar{\partial}\varphi)^{n-d}$ is closed. Let c_n denote its mass. We can construct a positive closed current T_{t_0} of mass 1 on $(\varphi = t_0)$ as a limit of $\frac{1}{c_n}\chi_n(\varphi)(i\partial\bar{\partial}\varphi)^{n-d}$. It follows also that when $d > 0$ there is a positive closed current S of bidimension (d, d) and mass 1 on ∂U . It suffices to take a cluster point of the currents T_t .

□

Remark 5.7. If $2d \geq n$, we have that $S \wedge S = 0$. If M is Kähler, then S is nef, i.e., it is a limit of smooth strictly positive closed forms.

Corollary 5.8. *Let (M, ω) be an infinitesimally homogeneous compact Kähler manifold. Let $U \subset M$ be a domain satisfying the Kontinuitätssatz. If $H_{dR}^2(U) = 0$, then U is Stein.*

Proof. Let T be a positive closed current of bidimension $(1, 1)$ with compact support in U . Let ω be a Kähler form. The cohomological hypothesis on the de Rham group, implies that we can write $\omega = d\alpha$, with α smooth. Then

$$\langle T, \omega \rangle = \langle T, d\alpha \rangle = -\langle dT, \alpha \rangle = 0.$$

So the dimension of the bundle \mathcal{N} is $d = 0$ and hence U is Stein.

□

Remark 5.9. Assume $U \Subset M$, with (M, ω) , compact Kähler, not infinitesimally homogeneous. Suppose $H_{dR}^2(U) = 0$, and that U admits a continuous psh exhaustion. If U is not Stein, then there are non-zero non-closed but $i\partial\bar{\partial}$ -closed currents with compact support in U . This follows from the argument in the previous corollary and from Theorem 1.5.

Corollary 5.10. *Suppose (M, U) are as in Theorem 5.5. Assume that M is a compact Kähler manifold. Let T_t be the positive closed current constructed in Theorem 5.5. Then $\{T_t\}^2 = 0$. If $d = n - 1$, all the currents T_t are nef and are in the same cohomology class.*

Proof. The currents T_t are directed by a foliation without singularities and they are closed. Then a result of Kaufman [19] states that the cohomology class $\{T_t\}$ of T_t satisfies $\{T_t\}^2 = 0$ provided $2d \geq n$.

When $d = n - 1$, $\{T_t\}^2 = \{T_{t'}\}^2 = 0$. If $t \neq t'$, since the supports of T_t and $T_{t'}$ are disjoint we get that $\{T_t\} \smile \{T_{t'}\} = 0$. As we have seen, $\{T_t\}$ and $\{T_{t'}\}$ are nef, then the Hodge-Riemann signature Theorem, implies that $\{T_t\}$ and $\{T_{t'}\}$ are proportional. If the two currents are on the same level set, we use a current on another level set.

□

Remark 5.11. Theorem 5.5 can be improved as follows. Suppose \mathcal{V} generates $T^{1,0}M$ at one point z_0 . Let A be the analytic set where $\text{rank}\mathcal{V}(z) \leq n-1$. Assume $U \not\subset \mathcal{V}$. Then there is a continuous psh exhaustion φ . There is also, an integer d , $0 \leq d < n$ and a foliation with leaves of dimension d on each $(\varphi = t) \setminus A$. In particular, if $d = 0$ and A is Stein, then U is Stein. If U is not Stein, then there is a nontrivial holomorphic image of \mathbb{C} , which is relatively compact in U . One shows if $d = 0$ that any positive $\partial\bar{\partial}$ -closed current has no mass out of A . So if A is Stein, it has also no mass on A . Hence, U is Stein by Theorem 1.5.

Remark 5.12. Let $U \Subset M$ be a pseudoconvex domain with smooth boundary in an infinitesimally homogeneous manifold. Assume it is not Stein and let d be the dimension of the leaves of the associated foliation \mathcal{F} . Then \mathcal{F} extends to a foliation on ∂U , with leaves of dimension d . This is a case where the dimension of the leaves does not change.

Corollary 5.13. *Let $U \subset M$ be as in Theorem 5.5. Assume ∂U is smooth. Assume that at a point $p \in \partial U$ the rank of the Levi form is maximal and equal to $n-l$. Then the dimension d of the foliation satisfies $d = l-1$. In particular, if p is a point of strict pseudoconvexity, then U is Stein.*

Proof. Suppose p is a point of strict pseudoconvexity i.e $l = 1$. Then there is a continuous psh function ψ in U : $\psi(p) = 0$, $\psi < 0$ on $B(p, r) \cap U$ and ψ strictly psh near p . Such a function implies that $d = 0$ and hence U is Stein. For the general case we can cut locally near p , by a subspace L_α of dimension $n-l+1$ such that $L_\alpha \cap \partial U$ is strictly pseudoconvex at p . Then the foliation induces on L_α leaves of dimension 0. Hence, the dimension of the original foliation is $d = l-1$. \square

Remark 5.14. When M is compact homogeneous and U has a point of strict pseudoconvexity, the result is due to Michel [20].

6 Levi-problem for Pfaff systems

Let (M, ω) be a complex hermitian manifold. Fix $\mathcal{S} = (\alpha_j)_{j \leq m}$ a Pfaff system, i.e., the α_j are $(1,0)$ -forms of class \mathcal{C}^1 on M . Assume that for every $z \in M$, $\bigcap_{j \leq m} \ker \alpha_j(z) \neq \{0\}$. We will say that M is \mathcal{S} -strongly pseudoconvex if it admits a smooth exhaustion u such that

$$\langle i\partial\bar{\partial}u(z), it_z \wedge \bar{t}_z \rangle > 0 \quad \text{for } t_z \neq 0, \quad \langle \alpha_j(z), t_z \rangle = 0, \quad 1 \leq j \leq m. \quad (6.1)$$

Let $U \Subset M$ be a domain with smooth boundary and defining function r . We will say that U is \mathcal{S} -pseudoconvex if

$$\langle i\partial\bar{\partial}u(z), it_z \wedge \bar{t}_z \rangle \geq 0 \quad \text{when } \langle \partial r(z), t_z \rangle = \langle \alpha_j(z), t_z \rangle = 0, \quad 1 \leq j \leq m. \quad (6.2)$$

The basic example of this situation is when the (α_j) are holomorphic and the system is integrable. Then we get a notion of uniform pseudoconvexity on leaves.

The question we address is assuming that U is \mathcal{S} -pseudoconvex in a complex manifold M , under which conditions is U \mathcal{S} -strongly pseudoconvex?

It is natural to introduce the cone \mathcal{P}_S of \mathcal{C}^2 -smooth functions in U such that $\langle i\partial\bar{\partial}v, it \wedge \bar{t} \rangle \geq 0$ when $\langle \alpha_j, t \rangle = 0$ for $1 \leq j \leq m$, see [27]. Denote by $\overline{\mathcal{P}}_S$ the space of continuous functions which are decreasing limits of functions in \mathcal{P}_S .

We can extend some of the results from previous sections to this context. As observed in [27], the estimates (4.2) and (4.3) are valid for positive $\partial\bar{\partial}$ -closed currents directed by \mathcal{S} , and for functions in $\overline{\mathcal{P}}_S$. A Levi current T for the system \mathcal{S} on U is a positive current of bidimension $(1, 1)$ such that

$$i\partial\bar{\partial}T = 0, \quad T \wedge \alpha_j = 0, \quad 1 \leq j \leq m, \quad T \wedge i\partial\bar{\partial}u = 0 \quad \text{for every } u \in \mathcal{P}_S.$$

A Levi current for the system \mathcal{S} on $(r = 0)$ satisfies

$$i\partial\bar{\partial}T = 0, \quad T \wedge i\partial\bar{\partial}r = 0, \quad T \wedge \partial r = 0, \quad T \wedge \alpha_j = 0, \quad 1 \leq j \leq m, \quad \langle T, \omega \rangle = 1. \quad (6.3)$$

We just state some extensions, leaving the proof to the reader.

Theorem 6.1. *Let $U \Subset M$ be an \mathcal{S} -pseudoconvex domain with smooth boundary. If there is no Levi current for the system \mathcal{S} on ∂U , then U is strongly \mathcal{S} -pseudoconvex. Moreover, there is a smooth function v , such that for A positive large enough and $\eta > 0$ small enough, $\hat{\rho} := -(re^{-Av})^\eta$ satisfies $i\partial\bar{\partial}\hat{\rho} \gtrsim C\omega|\hat{\rho}|$ on vectors such that $\langle \alpha_j(z), t_z \rangle = 0$, $1 \leq j \leq m$.*

Sketch of proof. One shows that if $T \geq 0$, $T \wedge \alpha_j = 0$ for $1 \leq j \leq m$, and $i\partial\bar{\partial}T = 0$, then $T \wedge \partial r = 0$ and $T \wedge i\partial\bar{\partial}r = 0$. So a duality argument implies that if there is no Levi current for the system \mathcal{S} on ∂U , there is a function $v \in \mathcal{P}$ which is strictly \mathcal{S} -psh, i.e., $\langle i\partial\bar{\partial}v(z), t_z \wedge \bar{t}_z \rangle > 0$ when $\langle \alpha_j, t_z \rangle = 0$ for $1 \leq j \leq m$. The proof then follows the lines of the proof of Theorem 2.2 using relations like

$$\langle i\partial\bar{\partial}r(z), t_z \wedge \bar{t}_z \rangle \geq -C|\langle \partial r(z), t \rangle||t| - C \sum_{j=1}^m |\langle \alpha_j(z), t \rangle||t|$$

for z in a neighborhood of ∂U . □

Remark 6.2. In [2] and [27], Hörmander type estimates, for $\bar{\partial}$ with respect to \mathcal{S} -directed currents with respect to \mathcal{S} -pseudoconvex functions as weights, are given.

Theorem 6.3. *Suppose $U \subset M$ admits a continuous exhaustion function which is \mathcal{S} -pseudoconvex. Then U admits a strictly \mathcal{S} -pseudoconvex exhaustion if and only if there is no Levi current, for the system \mathcal{S} , with compact support in U .*

The proof is an adaptation of the proof in Section 4. We omit it.

References

- [1] Berndtsson, Bo. L^2 -methods for the $\bar{\partial}$ -equation. preprint, available at <http://www.math.chalmers.se/~bob/>
- [2] Berndtsson, Bo; Sibony, Nessim. The $\bar{\partial}$ -equation on a positive current. *Invent. math.* **147** (2002), no. 2, 371-428.
- [3] Diederich, Klas; Fornaess, John Erik. Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions. *Invent. Math.* **39** (1977), no. 2, 129-141.
- [4] Diederich, Klas; Fornaess, John Erik. Pseudoconvex domains with real-analytic boundary. *Ann. Math. (2)* **107** (1978), no. 2, 371-384.
- [5] Diederich, Klas; Fornaess, John Erik. A smooth pseudoconvex domain without pseudoconvex exhaustion. *Manuscripta Math.* **39** (1982), no. 1, 119-123.
- [6] Diederich, Klas; Ohsawa, Takeo. A Levi problem on two-dimensional complex manifolds. *Math. Ann.* **261** (1982), no. 2, 255-261.
- [7] Donnelly, Harold; Fefferman, Charles. L^2 -cohomology and index theorem for the Bergman metric. *Ann. of Math. (2)* **118** (1983), no. 3, 593-618.
- [8] Dinh Tien-Cuong; Sibony Nessim. Pull-back currents by holomorphic maps, *Manuscripta Math.* **123** (2007), no. 3, 357-371.
- [9] Dinh Tien-Cuong; Sibony Nessim. Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings, 165-294, *Lecture Notes in Math.*, **1998**, Springer, Berlin, 2010.
- [10] Elencwajg, Georges. Pseudo-convexité locale dans les variété kählériennes. *Ann. Inst. Fourier (Grenoble)* **25** (1975), no. 2, xv, 295-314.
- [11] Fornaess, John Erik; Sibony, Nessim. Oka's inequality for currents and applications. *Math. Ann.* **301** (1995), no. 3, 399-419.
- [12] Fornaess, John Erik; Sibony, Nessim. Riemann surface laminations with singularities. *J. Geom. Anal.* **18** (2008), no. 2, 400-442.
- [13] Grauert, Hans. On Levi's problem and the imbedding of real-analytic manifolds. *Ann. of Math. (2)* **68** (1958) 460-472.
- [14] Grauert, Hans. Bemerkenswerte pseudokonvexe Mannigfaltigkeiten. *Math. Z.* **81** (1963) 377-391.
- [15] Grauert, H.; Kantenkohomologie. *Compositio Math.* **44** (1981), no. 1-3, 79-101.

- [16] Hirschowitz, André. Pseudoconvexité au-dessus d'espaces plus ou moins homogènes. *Invent. Math.* **26** (1974), 303-322.
- [17] Hirschowitz, André. Le problème de Lévi pour les espaces homogènes. *Bull. Soc. Math. France* **103** (1975), no. 2, 191-201.
- [18] Hörmander, Lars. An introduction to complex analysis in several variables, 3rd. ed. North Holland (1988).
- [19] Kaufmann, Lucas. Self-intersection of foliated cycles on complex manifolds. [arXiv:1602.07238](https://arxiv.org/abs/1602.07238).
- [20] Michel, Daniel. Sur les ouverts pseudo-convexes des espaces homogènes. *C. R. Acad. Sci. Paris Sér. A-B* **283** (1976), no. 10, 779-782.
- [21] Narasimhan, Raghavan. The Levi problem for complex spaces. II. *Math. Ann.* **146** (1962) 195-216.
- [22] Narasimhan, Raghavan. The Levi problem and pseudo-convex domains: a survey. *Enseign. Math. (2)* **24** (1978), no. 3-4, 161-172.
- [23] Ohsawa, Takeo; Sibony, Nessim. Bounded p.s.h. functions and pseudoconvexity in Kähler manifold. *Nagoya Math. J.* **149** (1998), 1-8.
- [24] Ohsawa, Takeo. Levi flat hypersurfaces. Results and questions around basic examples. **Preprint**.
- [25] Peternell, Thomas. Pseudoconvexity, the Levi problem and vanishing theorems. Complex analysis. Several variables, 7 Encyclopedia of Mathematical Sciences. Volume 74. Springer Verlag.
- [26] Richberg, Rolf. Stetige streng pseudokonvexe Funktionen. *Math. Ann.* **175** (1968) 257-286.
- [27] Sibony, Nessim. Pfaff systems, currents and hulls. *Math.Z.* (2016)
- [28] Sibony, Nessim. Quelques problèmes de prolongement de courants en analyse complexe, *Duke Math. J.*, **52** (1985), no.1, 157-197.
- [29] Siu, Yum Tong. Pseudoconvexity and the problem of Levi. *Bull. Amer. Math. Soc.* **84** (1978), no. 4, 481-512.
- [30] Skoda, Henri. Prolongement des courants positifs, fermés de masse finie, *Invent. Math.* **66** (1982), 361-376.
- [31] Sullivan, Dennis. Cycles for the dynamical study of foliated manifolds in complex manifolds. *Invent. Math.* **36**, 225-255, (1975).

- [32] Takeuchi, Akira. Domaines pseudoconvexes infinis et la métrique riemannienne dans un espace projectif. *J. Math. Soc. Japan* **16** (1964) 159-181.

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